

Singularly perturbed Neumann problem for fractional Schrödinger equations

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ABSTRACT. This paper is concerned with a Neumann type problem for singularly perturbed fractional nonlinear Schrödinger equations with subcritical exponent. For some smooth bounded domain $\Omega \subset \mathbf{R}^n$, our boundary condition is given by

$$\int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0 \quad \text{for } x \in \mathbf{R}^n \setminus \bar{\Omega}.$$

We establish existence of nonnegative small energy solutions, and also investigate the integrability of the solutions on \mathbf{R}^n .

1. Introduction

The main purpose of this paper is to investigate a singularly perturbed Neumann type problem for fractional Schrödinger equations. Precisely, given a smooth bounded domain $\Omega \subset \mathbf{R}^n$, we consider the following problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = |u|^{p-1}u & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{on } \mathbf{R}^n \setminus \bar{\Omega}. \end{cases} \quad (1.1)$$

Here $\varepsilon > 0$, $s \in (0, 1)$, $n \geq 2$, $p \in (1, \frac{n+2s}{n-2s})$, and

$$\mathcal{N}_s u(x) = C_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbf{R}^n \setminus \bar{\Omega}, \quad (1.2)$$

where $C_{n,s}$ is the normalization constant in the definition of fractional Laplacian

$$(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbf{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.$$

This type of boundary problem for fractional Laplacian was introduced by Dipierro, Ros-Oton and Valdinoci in [20]. It corresponds to a jump process as follows: If a particle has gone to $x \in \mathbf{R}^n \setminus \bar{\Omega}$, then it may come back to any point $y \in \Omega$, the probability of jumping from x to y being proportional to $|x - y|^{-n-2s}$. From mathematical point of

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view, such kind of boundary conditions generalize the classical Neumann conditions for elliptic (or parabolic) differential equations. That is, if $s \rightarrow 1$, then $\mathcal{N}_s u = 0$ becomes the classical Neumann condition. For more details, see [20]. Also in [21, 22], Du-Gunzburger-Lehoucq-Zhou introduced volume constraints for a general class of nonlocal diffusion problems on bounded domain in \mathbf{R}^n via a nonlocal vector calculus. If we rewrite (1.2) by a nonlocal vector calculus for fractional Laplacian, then $\mathcal{N}_s u = 0$ (with some modifications) can be considered as a special case of the volume constraints defined by [21, 22].

Other types of Neumann problems for fractional Laplacian (or other nonlocal operators) were investigated in many works [6, 13, 4, 5, 14, 15, 24]. All these conditions also have probabilistic interpretations and recover the classical Neumann problem as a limit case. A comparison between these models and ours can be found in [20, Section 7].

The singularly perturbed Neumann problem for classical nonlinear Schrödinger equations with subcritical exponent is as follows:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

where $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$ and $p > 1$ for $n = 2$, and ν is the unit outward normal on $\partial\Omega$. There is a great deal of works on this problem. We only restrict ourselves to cite a few papers, referring to the bibliography for further references. The pioneer works by Lin, Ni and Tagaki [32, 29, 33, 34] proved the existence of single-peak spike layer solution u_ε to (1.3). After that many interesting results concerning multi-peak spike-layer solutions to (1.3) have been obtained ([25, 26, 27]). Note that a spike-layer solution has its energy or mass concentrating near isolated points (a zero-dimensional set) in $\bar{\Omega}$. Similarly, there exist solutions to (1.3) with k -dimensional ($1 \leq k \leq n-1$) concentration set ([30, 31, 2, 3, 18]). We refer to [35] for more other results and references. Moreover, concentrating standing waves with a critical frequency for Schrödinger equations were obtained by [7, 8].

We should also mention that concentration phenomenon for fractional Schrödinger equations has been extensively studied recently. On the total space \mathbf{R}^n , the existence and multiplicity of spike layer solutions under various conditions were obtained by [17, 12, 11, 23]. On a bounded domain in \mathbf{R}^n , singularly perturbed Dirichlet problem was investigated by [16]. Moreover, under classical Neumann condition, an existence result of spike solutions to Schrödinger equations involving half Laplacian (see Equation (1.7) below) was proved by [38].

We are now in a position to formulate our main results and give the idea of the proofs. Our problem (1.1) has a variational structure. More precisely, let

$$\langle u, v \rangle_{H_{\varepsilon, \Omega}^s} = \frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} uv dx, \quad (1.4)$$

where $\Omega^c := \mathbf{R}^n \setminus \Omega$. Then define the space

$$H_{\varepsilon, \Omega}^s = \{u : \mathbf{R}^n \rightarrow \mathbf{R} \text{ measurable and } \langle u, u \rangle_{H_{\varepsilon, \Omega}^s} < \infty\}.$$

It is a Hilbert space with the norm $\|\cdot\|_{H_{\varepsilon, \Omega}^s} = \langle \cdot, \cdot \rangle_{H_{\varepsilon, \Omega}^s}^{\frac{1}{2}}$. It follows that weak solutions to the problem (1.1) are critical points of the following functional

$$I_\varepsilon(u) = \frac{C_{n,s}\varepsilon^{2s}}{4} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy + \frac{1}{2} \int_{\Omega} u^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx. \quad (1.5)$$

We obtain the following existence result.

THEOREM 1.1. *If ε is sufficiently small, then there exists a nonnegative weak solution u_ε to (1.1). Moreover, u_ε satisfies*

$$0 < I_\varepsilon(u_\varepsilon) \leq C_1 \varepsilon^n.$$

Consequently, u_ε is a nonconstant solution, and,

$$\|u_\varepsilon\|_{H_{\varepsilon, \Omega}^s} \leq C_2 \varepsilon^{\frac{n}{2}}. \quad (1.6)$$

Here C_1, C_2 are two positive constants depending only on n, s, p, Ω .

REMARK 1.2. The definition of weak solution is given by integrals (see Definition 2.1 below). Using nonlocal integration by parts formulas, we see that this definition is similar to the classical case (see Remark 2.5 below).

REMARK 1.3. Such kind of existence results for classical Neumann problem (1.3) was obtained by Lin, Ni and Takagi [29]. Recently, in [38], Stinga and Volzone recovered the results (including concentration and regularity issues) in [29] for a fractional semilinear Neumann problem as follows:

$$\begin{cases} \varepsilon(-\Delta)^{\frac{1}{2}}u + u = u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (1.7)$$

where $p \in (1, \frac{n+1}{n-1})$, ν is the outer unit normal to $\partial\Omega$. Note that both of the boundary conditions in (1.3) and (1.7) are classical.

The proof of Theorem 1.1 relies on critical point theory. More precisely, the functional I_ε has mountain pass structure. The key point is to construct an appropriate function $\phi \in H_{\varepsilon, \Omega}^s$ such that, for some $t_0 > 0$, it holds that $I_\varepsilon(t_0\phi) \leq 0$ and $0 < \sup_{t \in [0, t_0]} I_\varepsilon(t\phi) \leq C\varepsilon^n$ (C is a constant depending on n, s, Ω). As compared with the classical case, verifying the necessary properties of ϕ becomes more involved because of the fractional Laplacian. See Section 3 below.

Moreover, we investigate the integrability properties of the solutions to problem (1.1). We have the following theorem.

THEOREM 1.4. *Let $u \in H_{\varepsilon, \Omega}^s$. Then it holds that*

$$(1) \quad u \in L_{\text{loc}}^2(\mathbf{R}^n),$$

(2) if $\mathcal{N}_s(u) = 0$, then

$$\int_{\mathbf{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < +\infty. \quad (1.8)$$

REMARK 1.5. The second conclusion of this theorem implies that the nonnegative weak solution u_ε obtained in Theorem 1.1 is L_s integrable in the sense of Silvestre [37]. We should note that if $\mathcal{N}_s(u) = 0$, then $\lim_{|x| \rightarrow \infty} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx$ ([20, Proposition 3.13]). Therefore, in general, we can not expect to prove that u_ε is integrable on the total space \mathbf{R}^n . In particular, u_ε does not belong to the s -th order Sobolev space $H^s(\mathbf{R}^n)$. Hence it is too weak to obtain more regularity properties of u_ε (see for example [37, 9, 28]).

To prove Theorem 1.4, we need a detailed analysis of some singular integrals of the following form

$$u[A, B] := \int_A \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx,$$

where A and B are two measurable set in \mathbf{R}^n . Such kind of integral is also important in nonlocal minimal surfaces (see e.g. [10, 36]). Since $u \in H_{\varepsilon, \Omega}^s$, $u(\Omega, \Omega)$ and $u(\Omega, \Omega^c)$ are finite. Then we can prove the local integrability of u by choosing appropriate balls in \mathbf{R}^n . See Section 4 below.

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2. Variational structure

The Neumann problem (1.1) is variational.

Define the space

$$H_{\varepsilon, \Omega}^s = \{u : \mathbf{R}^n \rightarrow \mathbf{R} \text{ measurable and } \|u\|_{H_{\varepsilon, \Omega}^s} < \infty\}.$$

Here $\|\cdot\|_{H_{\varepsilon, \Omega}^s}$ is given by (1.4). We should note that constant functions $v(x) \equiv c$ on \mathbf{R}^n are contained in $H_{\varepsilon, \Omega}^s$.

REMARK 2.1. Let $H^s(\Omega)$ be the s -th Sobolev space in Ω with the norm given by

$$\|h\|_{H^s(\Omega)}^2 = \int_{\Omega} \int_{\Omega} \frac{|h(x) - h(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} h^2 dx.$$

(See, for example, [1], [19].) Therefore, if $u \in H_{\varepsilon, \Omega}^s$, then $u|_{\Omega}$ is in $H^s(\Omega)$. We define $H^s(\mathbf{R}^n)$ to be the usual s -th Sobolev space in \mathbf{R}^n with the norm

$$\|h\|_{H^s(\mathbf{R}^n)}^2 = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|h(x) - h(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\mathbf{R}^n} h^2 dx.$$

From (1.4), it holds that $H^s(\mathbf{R}^n) \subset H_{\varepsilon, \Omega}^s$.

LEMMA 2.2. $H_{\varepsilon, \Omega}^s$ is a Hilbert space with inner product given by (1.4).

PROOF. This lemma is the case $g = 0$ of Proposition 3.1 in [20]. We omit details of the proof here. \square

REMARK 2.3. From Remark 2.1 and Sobolev embedding $H^s(\Omega) \hookrightarrow L^q(\Omega)$ ($q \in (1, \frac{2n}{n-2s})$), we have that, for all $u \in H_{\varepsilon, \Omega}^s$,

$$\int_{\Omega} |u|^{p+1} dx < +\infty.$$

DEFINITION 2.4. We say that $u \in H_{\varepsilon, \Omega}^s$ is a weak solution to (1.1), if, for all $v \in H_{\varepsilon, \Omega}^s$, it holds that

$$\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} uv dx - \int_{\Omega} |u|^{p-1} uv dx = 0. \quad (2.1)$$

REMARK 2.5. A direct computation yields that for all $u, v \in C^2 \cap H_{\varepsilon, \Omega}^s$,

$$\frac{1}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} v(-\Delta)^s u dx + \int_{\Omega^c} v \mathcal{N}_s u dx. \quad (2.2)$$

This is corresponding to the classical Green's first identity. Thus the definition of weak solution is the same as the classical case. That is, from (2.2), (2.1) can formally become

$$\int_{\Omega} (\varepsilon^{2s} (-\Delta)^s u + u - |u|^{p-1} u) v dx + \varepsilon^{2s} \int_{\Omega^c} v \mathcal{N}_s u dx = 0.$$

PROPOSITION 2.6. Any critical point of I_{ε} (see (1.5)) is a weak solution of problem (1.1).

PROOF. For any $v \in H_{\varepsilon, \Omega}^s$, we have that

$$\begin{aligned} & I_{\varepsilon}(u + tv) \\ = & \frac{C_{n,s}\varepsilon^{2s}}{4} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|(u + tv)(x) - (u + tv)(y)|^2}{|x - y|^{n+2s}} dx dy \\ & + \int_{\Omega} (u + tv)^2 dx - \frac{1}{p+1} \int_{\Omega} |u + tv|^{p+1} dx \\ = & I_{\varepsilon}(u) \\ & + t \left(C_{n,s}\varepsilon^{2s} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} uv dx - \int_{\Omega} |u|^{p-1} uv dx \right) \\ & + t^2 \left(\frac{C_{n,s}\varepsilon^{2s}}{4} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} v^2 dx + p \int_{\Omega} |u + \theta_1 tv|^{p-1} v^2 dx \right), \end{aligned}$$

where $\theta_1 \in (0, 1)$. Thus

$$\begin{aligned} I'_{\varepsilon}(u)v &= \lim_{t \rightarrow 0} \frac{I_{\varepsilon}(u + tv) - I_{\varepsilon}(u)}{t} \\ &= C_{n,s}\varepsilon^{2s} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} uv dx - \int_{\Omega} |u|^{p-1} uv dx \\ &= \langle u, v \rangle_{H_{\varepsilon, \Omega}^s} - \int_{\Omega} |u|^{p-1} uv dx. \end{aligned} \quad (2.3)$$

Therefore, if u is a critical point of I_ε , then u is a weak solution to (1.1). \square

3. Proof of Theorem 1.1

LEMMA 3.1. I_ε satisfies Palais-Smale condition.

PROOF. Let $\{u_m\} \subset H_{\varepsilon,\Omega}^s$ be a Palais-Smale sequence such that $|I_\varepsilon(u_m)| \leq d$, for all $m \in \mathbb{N}$, is bounded and $I'_\varepsilon(u_m) \rightarrow 0$. Then

$$d \geq \left| \frac{1}{2} \|u_m\|_{H_{\varepsilon,\Omega}^s}^2 - \frac{1}{p+1} \int_{\Omega} |u_m|^{p+1} dx \right|. \quad (3.1)$$

Since $I'_\varepsilon(u_m) \rightarrow 0$, for any $\epsilon > 0$, there is an $M = M(\epsilon)$ such that for all $m \geq M$,

$$\begin{aligned} & |I'_\varepsilon(u_m)v| \\ &= \left| \left(\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{(u_m(x) - u_m(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_m v dx \right) \right. \\ & \quad \left. - \int_{\Omega} |u_m|^{p-1} u_m v dx \right| \\ &= \left| \langle u_m, v \rangle_{H_{\varepsilon,\Omega}^s} - \int_{\Omega} |u_m|^{p-1} u_m v dx \right| \leq \epsilon \|v\|_{H_{\varepsilon,\Omega}^s}, \end{aligned} \quad (3.2)$$

for all $v \in H_{\varepsilon,\Omega}^s$. If we choose $\epsilon = 1$, $v = u_m$, then (3.2) yields

$$\left| \int_{\Omega} |u_m|^{p+1} dx \right| \leq \|u_m\|_{H_{\varepsilon,\Omega}^s}^2 + \|u_m\|_{H_{\varepsilon,\Omega}^s}. \quad (3.3)$$

By (3.3) and (3.1), we have that

$$d \geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u_m\|_{H_{\varepsilon,\Omega}^s}^2 - \frac{1}{p+1} \|u_m\|_{H_{\varepsilon,\Omega}^s}.$$

Therefore, $\{u_m\}$ is bounded in $H_{\varepsilon,\Omega}^s$. Up to a subsequence, we assume that $u_m \rightharpoonup u$ in $H_{\varepsilon,\Omega}^s$. By Remark 2.1 and Sobolev embedding, $u_m \rightarrow u$ in $L^{p+1}(\Omega)$. So $|u_m|^{p-1} u_m \rightarrow |u|^{p-1} u$ in $L^{(p+1)/p}(\Omega)$. Equation (2.3) yields that

$$\begin{aligned} \|u_m - u\|_{H_{\varepsilon,\Omega}^s}^2 &= \langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u \rangle_{H_{\varepsilon,\Omega}^s} \\ &\quad + \int_{\Omega} (|u_m|^{p-1} u_m - |u|^{p-1} u) (u_m - u). \end{aligned}$$

Since $\{u_m\}$ is bounded, $u_m \rightharpoonup u$ in $H_{\varepsilon,\Omega}^s$ and $I'_\varepsilon(u_m) \rightarrow 0$, we have that

$$\langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u \rangle_{H_{\varepsilon,\Omega}^s} \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

By Hölder inequality, it holds that

$$\begin{aligned} & \left| \int_{\Omega} (|u_m|^{p-1} u_m - |u|^{p-1} u) (u_m - u) \right| \\ & \leq \| |u_m|^{p-1} u_m - |u|^{p-1} u \|_{L^{(p+1)/p}(\Omega)} \|u_m - u\|_{L^{p+1}(\Omega)} \rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$. Therefore, we have that $\|u_m - u\|_{H_{\varepsilon,\Omega}^s} \rightarrow 0$ as $m \rightarrow \infty$. It completes the proof. \square

LEMMA 3.2. *There exists a $\rho > 0$ such that $I_\varepsilon(u) > 0$ if $0 < \|u\|_{H_{\varepsilon,\Omega}^s} < \rho$ and $I_\varepsilon(u) \geq \beta > 0$ if $\|u\|_{H_{\varepsilon,\Omega}^s} = \rho$.*

PROOF. By Remark 2.1 and Sobolev embedding,

$$\int_{\Omega} |u|^{p+1} dx \leq \|u\|_{H_{\varepsilon,\Omega}^s}^{p+1}.$$

Since $p > 1$, the conclusion of this lemma holds. \square

LEMMA 3.3. *For sufficiently small $\varepsilon > 0$, there exist a nonconstant function $\phi \in H_{\varepsilon,\Omega}^s$ and positive constants t_0 such that $I_\varepsilon(t_0\phi) = 0$ and $I_\varepsilon(t\phi) \leq C\varepsilon^n$ if $t \in [0, t_0]$. Here C is a constant depending on n, s, Ω .*

To prove this lemma, we construct a special function. Without loss of generality, we assume that $0 \in \Omega$. Let $\varepsilon \in (0, 1)$ is small enough so that $B_{2\varepsilon} \subset \Omega$. Define

$$\phi(x) = \begin{cases} \varepsilon^{-n}(1 - \varepsilon^{-1}|x|) & \text{if } |x| < \varepsilon, \\ 0 & \text{if } |x| \geq \varepsilon. \end{cases}$$

LEMMA 3.4. *For sufficiently small ε , $\phi \in H_{\varepsilon,\Omega}^s$. Precisely, we have that*

$$\|\phi\|_{H_{\varepsilon,\Omega}^s}^2 \leq \frac{C}{\varepsilon^n}, \quad (3.4)$$

where C is a positive constant depending on n, s, p and Ω .

PROOF. A direct calculus yields

$$\begin{aligned} \int_{\Omega} \phi^2(x) dx &= \int_{B_\varepsilon} \frac{1}{\varepsilon^{2n}} \left(1 - \frac{|x|}{\varepsilon}\right)^2 dx \\ &= \frac{\omega_{n-1}}{\varepsilon^{2n}} \int_0^\varepsilon \left(1 - \frac{r}{\varepsilon}\right)^2 r^{n-1} dr = \left(\frac{2\omega_{n-1}}{n(n+1)(n+2)}\right) \frac{1}{\varepsilon^n}. \end{aligned} \quad (3.5)$$

Here ω_{n-1} is the area of unit sphere in \mathbf{R}^n . Thus it remains us to estimate

$$\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy. \quad (3.6)$$

Compute

$$\begin{aligned}
& \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \\
&= \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega^c} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \\
&\quad + \int_{\Omega} \int_{\Omega^c} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \\
&= \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy + 2 \int_{\Omega^c} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy.
\end{aligned} \tag{3.7}$$

Calculate

$$\begin{aligned}
& \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \\
&= \int_{B_\varepsilon} \int_{B_\varepsilon} \frac{\left| \frac{1}{\varepsilon^n} \left(1 - \frac{|x|}{\varepsilon} \right) - \frac{1}{\varepsilon^n} \left(1 - \frac{|y|}{\varepsilon} \right) \right|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{\left| \frac{1}{\varepsilon^n} \left(1 - \frac{|x|}{\varepsilon} \right) \right|^2}{|x - y|^{n+2s}} dx dy \\
&\quad + \int_{B_\varepsilon} \int_{\Omega \setminus B_\varepsilon} \frac{\left| \frac{1}{\varepsilon^n} \left(1 - \frac{|y|}{\varepsilon} \right) \right|^2}{|x - y|^{n+2s}} dx dy \\
&= \int_{B_\varepsilon} \int_{B_\varepsilon} \frac{\left| \frac{1}{\varepsilon^n} \left(1 - \frac{|x|}{\varepsilon} \right) - \frac{1}{\varepsilon^n} \left(1 - \frac{|y|}{\varepsilon} \right) \right|^2}{|x - y|^{n+2s}} dx dy + 2 \int_{\Omega \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{\left| \frac{1}{\varepsilon^n} \left(1 - \frac{|x|}{\varepsilon} \right) \right|^2}{|x - y|^{n+2s}} dx dy. \\
&:= T_1 + T_2.
\end{aligned} \tag{3.8}$$

Estimating T_1 , we have that

$$\begin{aligned}
T_1 &= \frac{1}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{B_\varepsilon} \frac{||x| - |y||^2}{|x - y|^{n+2s}} dx dy \\
&\leq \frac{1}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{B_\varepsilon} \frac{|x - y|^2}{|x - y|^{n+2s}} dx dy = \frac{1}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{B_\varepsilon} \frac{1}{|x - y|^{n+2s-2}} dx dy \\
&\leq \frac{1}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{B_{2\varepsilon}(y)} \frac{1}{|x - y|^{n+2s-2}} dx dy = \frac{\omega_{n-1}}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \left\{ \int_0^{2\varepsilon} \frac{1}{r^{n+2s-2}} r^{n-1} dr \right\} dy \\
&\leq \frac{C}{\varepsilon^{n+2s}}.
\end{aligned}$$

And calculate T_2 :

$$\begin{aligned}
T_2 &= \frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dx dy \\
&= \frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_{2\varepsilon}} \int_{B_\varepsilon} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dx dy + \frac{2}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dx dy.
\end{aligned}$$

Then estimate

$$\begin{aligned}
& \frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_{2\varepsilon}} \int_{B_\varepsilon} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dx dy \\
&= \frac{2}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{\Omega \setminus B_{2\varepsilon}} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dy dx \leq \frac{2}{\varepsilon^{2n+2}} \int_{B_\varepsilon} \int_{\Omega \setminus B_\varepsilon} \frac{\varepsilon^2}{|y|^{n+2s}} dy dx \\
&\leq \frac{2\omega_{n-1}}{\varepsilon^{2n}} \int_{B_\varepsilon} \left\{ \int_\varepsilon^{+\infty} \frac{1}{r^{n+2s}} r^{n-1} dr \right\} dx = \frac{\omega_{n-1}}{s\varepsilon^{2n+2s}} \int_{B_\varepsilon} dx \\
&\leq \frac{C}{\varepsilon^{n+2s}},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{2}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} dx dy \\
&\leq \frac{2}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{(|y| - |x|)^2}{|x - y|^{n+2s}} dx dy \leq \frac{2}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \int_{B_\varepsilon} \frac{1}{|x - y|^{n+2s-2}} dx dy \\
&\leq \frac{2}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \int_{B_{3\varepsilon}(y)} \frac{1}{|x - y|^{n+2s-2}} dx dy = \frac{2\omega_{n-1}}{\varepsilon^{2n+2}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} \left\{ \int_0^{3\varepsilon} \frac{1}{r^{n+2s-2}} r^{n-1} dr \right\} dy \\
&= \frac{2 \cdot 3^{2-2s} \omega_{n-1}}{\varepsilon^{2n+2s}} \int_{B_{2\varepsilon} \setminus B_\varepsilon} dx = \frac{2 \cdot 3^{2-2s} (2^n - 1) \omega_{n-1}^2}{n\varepsilon^{n+2s}}.
\end{aligned}$$

Therefore, we obtain that

$$\int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy = T_1 + T_2 \leq \frac{C}{\varepsilon^{n+2s}}, \quad (3.9)$$

where C is a positive constant depending on n, s, Ω .

Finally, from $B_{2\varepsilon} \subset \Omega$, we have

$$\begin{aligned}
& 2 \int_{\Omega^c} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \\
&= 2 \int_{\Omega^c} \int_{B_\varepsilon} \frac{\frac{1}{\varepsilon^{2n}} \left(1 - \frac{|x|}{\varepsilon}\right)^2}{|x - y|^{n+2s}} dx dy \leq \frac{C}{\varepsilon^{2n}} \int_{\Omega^c} \int_{B_\varepsilon} \frac{\left(1 - \frac{|x|}{\varepsilon}\right)^2}{\left|\frac{y}{2}\right|^{n+2s}} dx dy \\
&\leq \frac{C}{\varepsilon^{2n}} \int_{B_{2\varepsilon}^c} \int_{B_\varepsilon} \frac{\left(1 - \frac{|x|}{\varepsilon}\right)^2}{\left|\frac{y}{2}\right|^{n+2s}} dx dy \leq \frac{C}{\varepsilon^{n+2s}}. \quad (3.10)
\end{aligned}$$

Summarizing the estimates (3.6), (3.7), (3.8), (3.9) and (3.10), we have that

$$\frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy \leq \frac{C}{\varepsilon^n}, \quad (3.11)$$

where C is a positive constant depending on n, s, Ω . From (3.5) and (3.11), we obtain (3.4). This completes the proof. \square

Define

$$g(t) = I_\varepsilon(t\phi), \quad t \geq 0.$$

LEMMA 3.5. *Assume that $\varepsilon > 0$ sufficiently small, there exist t_1 and t_2 with $0 < t_1 < t_2$ such that*

- (1) for $t > t_1$, $g'(t) < 0$;
- (2) for $t > t_2$, $g(t) < 0$.

PROOF. By a similar argument as in the proof of Lemma 3.4, we obtain that, for $t > 0$,

$$\frac{C_{n,s}\varepsilon^{2s}}{4} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|t\phi(x) - t\phi(y)|^2}{|x - y|^{n+2s}} dx dy + \frac{1}{2} \int_{\Omega} (t\phi)^2 dx \leq C_0 t^2 \varepsilon^{-n},$$

where C_0 is a positive constant depending on n, s, Ω . Moreover,

$$\begin{aligned} \int_{\Omega} (t\phi)_+^{p+1} dx &= \frac{t^{p+1}}{\varepsilon^{n(p+1)}} \int_{B_\varepsilon} \left(1 - \frac{|x|}{\varepsilon}\right)^{p+1} dx \\ &= \frac{\omega_{n-1} t^{p+1}}{\varepsilon^{n(p+1)}} \int_0^\varepsilon \left(1 - \frac{r}{\varepsilon}\right)^{p+1} r^{n-1} dr \\ &= \frac{\omega_{n-1} t^{p+1}}{\varepsilon^{np}} \int_0^1 (1 - \rho)^{p+1} \rho^{n-1} d\rho = \frac{\alpha \omega_{n-1} t^{p+1}}{\varepsilon^{np}}, \end{aligned}$$

where $\alpha := \int_0^1 (1 - \rho)^{p+1} \rho^{n-1} d\rho$. Let $t_2 = \left(\frac{C_0(p+1)}{\alpha \omega_{n-1}}\right)^{\frac{1}{p-1}} \varepsilon^n$. Then for all $t > t_2$, it holds that $g(t) = I_\varepsilon(t\phi) < 0$.

Next compute

$$\begin{aligned} g'(t) &= \frac{C_{n,s}\varepsilon^{2s}t}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dx dy + t \int_{\Omega} \phi^2 dx - t^p \int_{\Omega} \phi^{p+1} dx \\ &\leq 2C_0 t \varepsilon^{-n} - \frac{\alpha \omega_{n-1} t^p}{\varepsilon^{np}}. \end{aligned}$$

Let $t_1 = \left(\frac{2C_0}{\alpha \omega_{n-1}}\right)^{\frac{1}{p-1}} \varepsilon^n$. Thus if choosing $t > t_1$, we have that $g'(t) < 0$. Since $p > 1$, it holds that $t_1 < t_2$. This completes the proof. \square

PROOF OF LEMMA 3.3. By Lemma 3.2, it holds that $g(t) > 0$ if t is positive and sufficiently small. Then from Lemma 3.5, we have that there exists a $t_0 > 0$ such that

$g(t_0) = 0$. Moreover, estimate

$$\begin{aligned} \max_{t \geq 0} g(t) &= \max_{0 \leq t \leq t_1} g(t) \\ &\leq \max_{0 \leq t \leq t_1} \left\{ C_0 t^2 \varepsilon^{-n} - \frac{1}{p+1} \int_{\Omega} (t\phi)^{p+1} dx \right\} \\ &\leq \max_{0 \leq t \leq t_1} C_0 t^2 \varepsilon^{-n} \\ &= C_0 t_1^2 \varepsilon^{-n} = C_1 \varepsilon^n, \end{aligned}$$

where $C_1 = C_0 \left(\frac{2C_0}{\alpha \omega_{n-1}} \right)^{\frac{2}{p-1}}$. It completes the proof. \square

PROOF OF THEOREM 1.1. (1) Let $e = t_0 \phi$ and $\Gamma = \{\gamma \in C([0, 1], H_{\varepsilon, \Omega}^s) \mid \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. Then by Mountain Pass Theorem, we have that

$$c_{\varepsilon} := \inf_{\gamma \in \Gamma} \sup_{s \in [0, 1]} I_{\varepsilon}(\gamma(s)) > 0$$

is a critical value of I_{ε} . Then there exists a critical point u_{ε} such that

$$I_{\varepsilon}(u_{\varepsilon}) = c_{\varepsilon} \leq C_1 \varepsilon^n.$$

Note that the unique constant solution to (1.1) is $u \equiv 1$ on \mathbf{R}^n . A direct calculate yields

$$I_{\varepsilon}(1) = \left(\frac{1}{2} - \frac{1}{p+1} \right) |\Omega| > 0,$$

where $|\Omega|$ denotes the volume of Ω . Thus for ε small enough, we have that

$$I_{\varepsilon}(u_{\varepsilon}) < I_{\varepsilon}(1).$$

Therefore, u_{ε} is a nonconstant solution to (1.1).

(2) Since u_{ε} is a solution to (1.1), we have that

$$\frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u_{\varepsilon}(x) - u_{\varepsilon}(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_{\varepsilon}^2 dx = \int_{\Omega} |u_{\varepsilon}|^{p+1} dx. \quad (3.12)$$

Then by the definition of I_{ε} , it holds that

$$\begin{aligned} &I_{\varepsilon}(u_{\varepsilon}) \\ &= \frac{1}{2} \left(\frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u_{\varepsilon}(x) - u_{\varepsilon}(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_{\varepsilon}^2 dx \right) - \frac{1}{p+1} \int_{\Omega} |u_{\varepsilon}|^{p+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u_{\varepsilon}(x) - u_{\varepsilon}(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_{\varepsilon}^2 dx \right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u_{\varepsilon}\|_{H_{\varepsilon, \Omega}^s}^2. \end{aligned}$$

Thus, choosing $C_2 = 2C_1(\frac{p+1}{p-1})$, we obtain (1.6).

(3) We prove that there exists a critical point $u_\varepsilon \geq 0$ in \mathbf{R}^n . In fact, when $u_\varepsilon \leq 0$, we can choose $-u_\varepsilon$. Thus we only need to exclude the sign change case. We argue by contradiction. Assume that u_ε is a sign change critical point obtained above. Note that

$$(|u_\varepsilon(x)| - |u_\varepsilon(y)|)^2 \leq |u_\varepsilon(x) - u_\varepsilon(y)|^2.$$

Then we have the following Kato-type inequality

$$\int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(|u_\varepsilon(x)| - |u_\varepsilon(y)|)^2}{|x - y|^{n+2s}} dx dy < \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{n+2s}} dx dy. \quad (3.13)$$

The strict inequality is from that u_ε is sign change. It follows that

$$I_\varepsilon(|u_\varepsilon|) < I_\varepsilon(u_\varepsilon). \quad (3.14)$$

Let

$$\begin{aligned} f(t) &= I_\varepsilon(t|u_\varepsilon|) \\ &= \frac{t^2}{2} \left(\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(|u_\varepsilon(x)| - |u_\varepsilon(y)|)^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_\varepsilon^2 dx \right) - \frac{t^{p+1}}{p+1} \int_{\Omega} |u_\varepsilon|^{p+1} dx, \end{aligned}$$

where $t \in [0, +\infty)$. For simplicity, set

$$\Lambda := \left(\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_\varepsilon^2 dx \right) > 0,$$

$$\bar{\Lambda} := \left(\frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbf{R}^{2n} \setminus (\Omega^c)^2} \frac{(|u_\varepsilon(x)| - |u_\varepsilon(y)|)^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega} u_\varepsilon^2 dx \right) > 0,$$

and

$$\Xi := \int_{\Omega} |u_\varepsilon|^{p+1} dx > 0.$$

Thus

$$f(t) = \frac{\bar{\Lambda}}{2} t^2 - \frac{\Xi}{p+1} t^{p+1},$$

and

$$\bar{\Lambda} < \Lambda.$$

Since u_ε is critical point, (3.12) yields

$$\Lambda = \Xi \quad \text{and} \quad \frac{\Lambda}{2} - \frac{\Xi}{p+1} = c_\varepsilon.$$

Note that $f(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. f has a unique maximum point at $t_1 = (\frac{\bar{\Lambda}}{\Xi})^{\frac{1}{p-1}}$ and

$$\begin{aligned} f(t_1) &= \frac{\bar{\Lambda}}{2} \left(\frac{\bar{\Lambda}}{\Xi} \right)^{\frac{2}{p-1}} - \frac{\Xi}{p+1} \left(\frac{\bar{\Lambda}}{\Xi} \right)^{\frac{p+1}{p-1}} \\ &= \left(\frac{\bar{\Lambda}}{\Xi} \right)^{\frac{2}{p-1}} \left(\frac{\bar{\Lambda}}{2} + \left(c - \frac{\Lambda}{2} \right) \left(\frac{\bar{\Lambda}}{\Xi} \right) \right) \\ &= \left(\frac{\bar{\Lambda}}{\Xi} \right)^{\frac{2}{p-1}} c_\varepsilon < c_\varepsilon. \end{aligned}$$

Thus, there exists $t_2 > 0$ such that $f(t_2) = 0$. Let $\bar{e} = t_2 u_\varepsilon$. We now set $V^+ = \{\lambda e + \mu \bar{e} \mid \lambda \geq 0, \mu \geq 0\}$. Therefore, there is an $R_0 > \max\{\|e\|_{H_{\varepsilon,\Omega}^s}, \|\bar{e}\|_{H_{\varepsilon,\Omega}^s}\}$ such that for all $u \in V^+$ with $\|u\|_{H_{\varepsilon,\Omega}^s} \geq R_0$, it holds that $I_\varepsilon(u) < 0$. Let γ_0 be the path which consists of the line segment with endpoints 0 and $R_0 \bar{e} / \|\bar{e}\|_{H_{\varepsilon,\Omega}^s}$, the circular arc $\partial B_{R_0} \cap V^+$, and the line segment with endpoints $R_0 e / \|e\|_{H_{\varepsilon,\Omega}^s}$ and e . Hence γ_0 belongs to Γ . However, along γ_0 , I_ε is positive only on the line joining 0 and \bar{e} . This yields that

$$\sup_{u \in \gamma_0} I_\varepsilon(u) = f(t_1) < c_\varepsilon.$$

It is a contradiction. Therefore, there is a critical point $u_\varepsilon \geq 0$. This completes the proof. \square

4. Proof of Theorem 1.4

PROOF OF THEOREM 1.4. (1) Without loss of generality, we assume that $0 \in \Omega$. Then there exists $R_0 > 0$ such that $B_{2R_0} \subset \Omega$. Since $u \in H_{\varepsilon,\Omega}^s$, it holds that

$$I := \int_{\Omega} \int_{\Omega^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx < +\infty.$$

Particularly,

$$\int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx \leq I < +\infty.$$

Here $\rho > 0$ is constant and $\Omega_\rho = \{y \in \mathbf{R}^n \mid d(y, \Omega) < \rho\}$. A direct computation yields

$$\begin{aligned} & \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx \\ & \geq \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x)|^2}{|x - y|^{n+2s}} dy dx + \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(y)|^2}{|x - y|^{n+2s}} dy dx \\ & \quad - 2 \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x)u(y)|}{|x - y|^{n+2s}} dy dx := T_1 + T_2 - T_3. \end{aligned}$$

We now estimate these three terms. Firstly,

$$\begin{aligned} T_1 &= \int_{B_{R_0}} |u(x)|^2 \left\{ \int_{\Omega_\rho \cap \Omega^c} \frac{1}{|x-y|^{n+2s}} dy \right\} dx \\ &\geq \frac{|\Omega_\rho \cap \Omega^c|}{(d(\Omega) + \rho)^{n+2s}} \int_{B_{R_0}} |u(x)|^2 dx := a. \end{aligned}$$

where $d(\Omega)$ denotes the diameter of Ω and $|\Omega_\rho \cap \Omega^c|$ is the volume of $\Omega_\rho \cap \Omega^c$. Note that $\int_{B_{R_0}} |u(x)|^2 dx < \infty$ since $\|u\|_{H^s(\Omega)} \leq \|u\|_{H^s_{\varepsilon,\Omega}}$. Thus a is a nonnegative constant depending on Ω, R_0, ρ, n, s . Secondly,

$$\begin{aligned} T_2 &\geq \frac{1}{(d(\Omega) + \rho)^{n+2s}} \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} |u(y)|^2 dy dx \\ &= \frac{|B_{R_0}|}{(d(\Omega) + \rho)^{n+2s}} \int_{\Omega_\rho \cap \Omega^c} |u(y)|^2 dy \\ &:= b \int_{\Omega_\rho \cap \Omega^c} |u(y)|^2 dy. \end{aligned}$$

Here b is a positive constant depending on Ω, R_0, ρ, n, s . Finally,

$$\begin{aligned} T_3 &= 2 \int_{B_{R_0}} |u(x)| \int_{\Omega_\rho \cap \Omega^c} \frac{|u(y)|}{|x-y|^{n+2s}} dy dx \\ &\leq \frac{2}{d(B_{R_0}, \partial\Omega)^{n+2s}} \int_{B_{R_0}} |u(x)| \left\{ \int_{\Omega_\rho \cap \Omega^c} |u(y)| dy \right\} dx \\ &= \frac{2 \int_{B_{R_0}} |u(x)| dx}{d(B_{R_0}, \partial\Omega)^{n+2s}} \int_{\Omega_\rho \cap \Omega^c} |u(y)| dy \\ &:= c \int_{\Omega_\rho \cap \Omega^c} |u(y)| dy, \end{aligned}$$

where c is also a nonnegative constant depending on Ω, R_0, ρ, n, s . Therefore, we obtain that

$$I \geq a + b \int_{\Omega_\rho \cap \Omega^c} |u(y)|^2 dy - c \int_{\Omega_\rho \cap \Omega^c} |u(y)| dy. \quad (4.1)$$

Note that by the proof of (4.1), it holds that, for any $X \subset \Omega_\rho \cap \Omega^c$,

$$I \geq a + b \int_X |u(y)|^2 dy - c \int_X |u(y)| dy.$$

We then argue by contradiction. Assume that $u \notin L^2(\Omega_\rho \cap \Omega^c)$, that is

$$\int_{\Omega_\rho \cap \Omega^c} |u(y)|^2 dy = +\infty.$$

From (4.1), we have that $u \notin L^1(\Omega_\rho \cap \Omega^c)$. Let

$$A_k := \{y \in \Omega_\rho \cap \Omega^c \mid |u(y)| > 2^k\}$$

and

$$D_k := A_k \setminus A_{k+1} = \{y \in \Omega_\rho \cap \Omega^c \mid 2^k < |u(y)| \leq 2^{k+1}\}.$$

Set d_k to be the measure of D_k . Let N_1 be a positive integer such that $2^{N_1-1} > \frac{c}{b}$. Then, for all $N_2 > N_1$,

$$\int_{A_{N_1} \setminus A_{N_2}} |u(y)| dy = \sum_{k=N_1}^{N_2} \int_{A_k \setminus A_{k+1}} |u(y)| dy \leq \sum_{k=N_1}^{N_2} 2^{k+1} d_k \rightarrow +\infty, \quad \text{as } N_2 \rightarrow \infty, \quad (4.2)$$

and

$$\int_{A_{N_1} \setminus A_{N_2}} |u(y)|^2 dy = \sum_{k=N_1}^{N_2} \int_{A_k \setminus A_{k+1}} |u(y)|^2 dy \geq \sum_{k=N_1}^{N_2} 2^{2k} d_k \rightarrow +\infty, \quad \text{as } N_2 \rightarrow \infty.$$

Since

$$\sum_{k=N_1}^{N_2} 2^{2k} d_k > 2^{N_1-1} \sum_{k=N_1}^{N_2} 2^{k+1} d_k,$$

we have that

$$\begin{aligned} I &\geq a + b \int_{A_{N_1} \setminus A_{N_2}} |u(y)|^2 dy - c \int_{A_{N_1} \setminus A_{N_2}} |u(y)| dy \\ &> a + 2^{N_1-1} b \sum_{k=N_1}^{N_2} 2^{k+1} d_k - c \sum_{k=N_1}^{N_2} 2^{k+1} d_k. \end{aligned}$$

It is a contradiction to (4.2). Therefore, $u \in L^2(\Omega_\rho \cap \Omega^c)$. Note that $u \in L^2(\Omega)$, we obtain that $u \in L^2(\Omega_\rho)$. Since ρ is arbitrary, it follows that $u \in L^2_{\text{loc}}(\mathbf{R}^n)$.

(2) Let R be a positive constant such that $\Omega \subset B_R(0)$. By the proof of Proposition 3.13 in [20], we know that if $\mathcal{N}_s(u) = 0$, then u is bounded in $B_R^c(0)$ with

$$\lim_{|x| \rightarrow \infty} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx \quad \text{uniformly in } x.$$

Therefore,

$$\int_{B_R^c(0)} \frac{|u(x)|}{1 + |x|^{n+2s}} dx \leq C \int_{B_R^c(0)} \frac{1}{1 + |x|^{n+2s}} dx < +\infty,$$

where C is a constant such that $\sup_{B_R^c(0)} |u| \leq C$. So, we only need to consider u on $B_R(0)$. From the conclusion (1), it follows that

$$u|_{B_R(0)} \in L^2(B_R(0)).$$

Thus, we have that

$$\int_{B_R(0)} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < +\infty.$$

This completes the proof. \square

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